Collatz Conjecture Session Guide

Session Outline

Description

In this session, learners will explore an unsolved mathematical problem that was initially presented in 1937 by Lothar Collatz. While famous mathematician Paul Erdos stated that “mathematics is not ready for this kind of problems”, it is simple enough for a young child to understand and play with. In short, you start with a positive integer. If it is even you divide it by 2 and if it’s odd you multiply it by 3 and add 1. You then repeat this for the result you get, and keep repeating for subsequent results. The problem suggests that regardless of what number you start with you will eventually get to number 1. For example, starting 7 it would go as follows:
7 -> 3x7+1=22 -> 22/2=11 -> 3x11+1=34 -> 34/2=17 -> 3x17+1=52 -> 52/2=26 -> 26/2=13 -> 3x13+1=40 -> 40/2=20 -> 20/2=10-> 10/2=5 -> 3x5+1=16 -> 16/2=8 -> 8/2=4 -> 4/2=2 -> 2/2=1.
While an incredibly large number of starting numbers were tested so far, there is no certainty that we will always get to the number 1. This session presents this problem in a fun and engaging way, bringing in competition and collaboration.

Session Objectives

The objectives of this session are to:

- Be able to carry out mathematical processes accurately.
- To carry out simple calculations repeatedly.
- To reach conclusions based on evidence.
- To understand the importance of justification and be aware that there are problems that are yet to be solved.

Expected Outcomes

By the end of the session students will have:
- Worked with number patterns.
- Recognised links between solutions to problems.
- Organised data to present a solution graphically.

Areas involved

- Mathematics and Financial Literacy
  - Math brain teasers
  - The Number System
- Environment
  - Careers in Science
- Life Skills
  - Self-esteem

Understanding the Problem

Objectives

To understand how the sequence works and be able to apply it to small numbers.

Expected Outcomes

Learners should be able to apply the process to small numbers to test that they always get to the number 1. They should also understand that when they get to 1, the cycle 4, 2, 1, 4, 2, 1, 4, 2, 1… keeps repeating forever and we can consider the process finished when we get to 1.

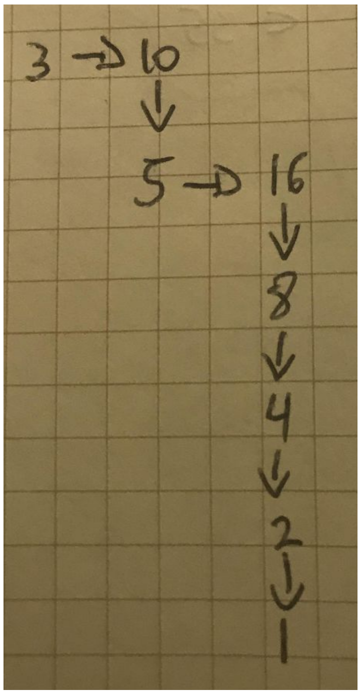
Teaching Instructions

- Explain the problem by asking a learner to select a number between 5 and 10. Ask another learner to divide it by 2 if it's even or multiply it by 3 and add 1 if it's odd. Ask the same question with the result to another learner. Keep going until you get to 1, and keep going through the cycle to demonstrate that once you get to 1 you will simply keep repeating the same three numbers.
- Repeat the process in the same way with another starting number between 5 and 10 until you get to 1 and let learners decide that you can now stop.
- Pose the question of whether it will always get to 1. Allow learners to try it out with a couple of numbers between 10 and 20 and after a while ask individuals what number they started with and whether they got to 1..

Suggested Guidelines

- Suggested time: 10-15 minutes.

- This activity is ideal to engage all students even if it is one at a time, with simple tasks that they should be able to achieve and build confidence.
- If you have access to a board or a flipchart it would be great to depict sequences with a starting number and an arrow going to the next one, as in the following picture demonstrating the case starting with 3 (with arrows going down when you divide by 2 and across when you multiply by 3 and add 1). This will be helpful later on in the session, so doing it from the start might be useful.



Activity

- Pick a number from 5 to 10. If it's even, divide it by 2, if it's odd multiply it by 3 and add 1. If the result is an even number, divide it by 2 again. If it's odd, multiply it by 3 and add 1. Keep repeating the process until you can predict what will happen.
- Conclude that you can stop when you get to 1.
- Pick another number from 5 to 10 and repeat the process. Do you also get to 1?
- Now repeat the process for a few more numbers from 10 to 20. Do you also get to 1?

Competition: The Longest Sequence

Objectives

- To apply the process accurately and confidently.

- To realise that larger numbers don't always lead to longer sequences.
- To reinforce that the conjecture holds for 'small' numbers.

Expected outcomes

Learners will apply the sequence to more numbers in order to find longer sequences. Learners will feel comfortable in their understanding of how the sequence works and become more convinced that they will always get to 1.

Teaching Instructions

- Explain the meaning of the length of the sequence: the number of terms you have to go through in order to get to 1, including the starting number and 1 itself. (for example, the length of the sequence starting with 3, i.e. 3, 10, 5, 16, 8, 4, 2, 1, has length 8, and the length of the sequence starting with 7, i.e. 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, has length 17).
- Introduce the competition. Individually, learners will work with numbers up to 64 for a given period of time, trying to find the longest sequence they can.
- To find the winner, students suggest the length of the longest sequence they could find and the number it started from. Other students verify that the length is correct and if it's not the next longest sequence can be considered for the winner.

Suggested Guidelines

- Recommended time: 15-20 minutes.
- Make it clear how much time learners will have to find the longest sequence and that when the time is up they have to stop and they cannot keep counting how many terms they have.
- Remind students when they have 3 minutes left and when they have 1 minute left.
- Encourage students to be organised, using the suggested diagrams with arrows presented in the previous activity. You can ask if the arrow diagrams would be of any help (and it can be, for once you get to a number in a diagram you already wrote the rest of the diagram will follow in exactly the same way). Don't make this fact explicit, it would be best if learners discover this, and it is the focus of the next activity.
- To engage more students and add a bit of fun and suspense, you can start with the potential third place, then the second and then the first (in each case confirming the position depending on the correct calculations in the sequence).

Activity

The length of a sequence is the number of terms it has. If we finish sequences when we reach the number 1, then the number of terms would represent the length of the sequence. For example, starting with 7 we get the sequence 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, which has length 17 as it has 17 terms.

We will have a competition to see who can get the longest sequence with numbers up to and including 64. The winner is the person who can get the longest sequence of the group.

Drawing Sequences on Trees

Objectives

- To recognise that all numbers from 1 to 20, 30, 40 or any larger number (time dependent) lead to 1.
- To conclude that it is likely that all numbers lead to 1.
- To be able to represent solutions in an organised manner and link solutions for different numbers.

Expected Outcomes

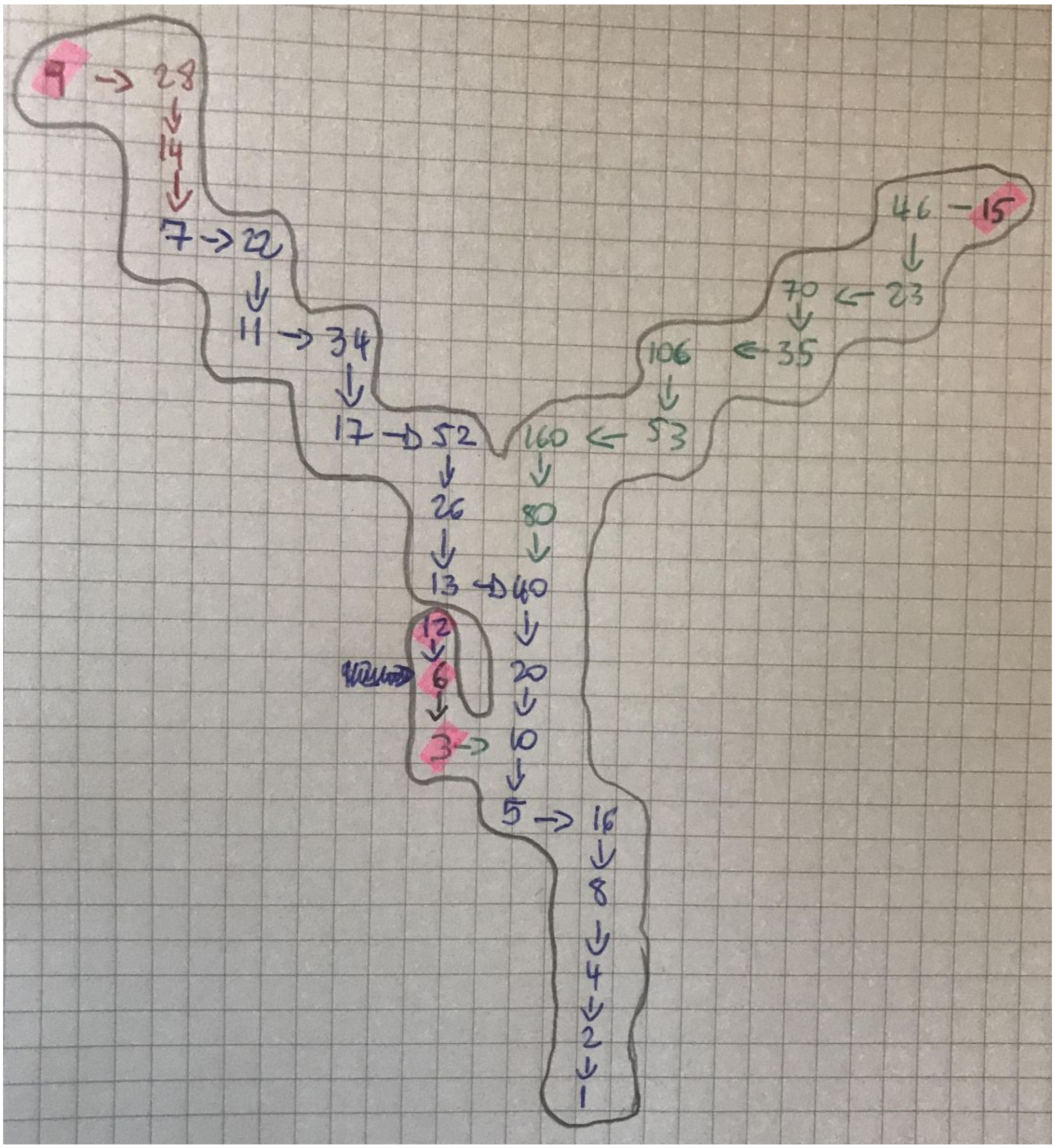
- Learners will draw trees to outline their solutions.
- Learners should be able to recognise how the solution for one number can help get the solution for another number.

Teacher Instructions

- Explain the method for representing answers graphically with a diagram in which you go down when you divide by 2 and you go to the right when you multiply by 3 and add 1.
- Starting from 1, check what numbers are in the diagram until you find a number that is not.
- Work out the sequence for the first missing number until you get to a number that is already in the diagram and add it.
- Find the next number that is not in the diagram and work out the sequence until you reach a number that is in the diagram and add it to the tree (this might start to get complicated but try to fit it in, perhaps going to the left instead of going to the right as before).
- Repeat a few more times for the next missing numbers.
- Let students keep going.

- Conclude that you get a tree shaped diagram where you can draw it in such a way that branches do not cross each other.
- Pose the question of whether this can help to find long sequences.

Below is a picture of a tree constructed by starting with number 7, and then adding the missing numbers 3, 6, 9, 12 and 15 (with their corresponding sequences) until their sequences link with the existing tree diagram.



Suggested Guidelines

- Suggested time: 15-20 minutes.
- Recommend learners to leave space above and to the sides of their initial diagram.
- Recommend that learners work on a separate piece of paper for each number and then plan how to link them so that branches don't cross each other.

Activity

- You can represent your sequences graphically where you start with your initial number, go down if you divide by 2 and to the right if you multiply by 3 and add 1, ending your diagram when you reach 1.
- Start counting from 1 until you find the first number that doesn't appear in your diagram. Work out the sequence for that number and link it to your initial diagram.
- Keep counting until you find a second number, work out the sequence and add it to the diagram.
- Continue counting and adding new sequences for missing numbers. Try to avoid crossing by writing smaller or going to the left or diagonally rather than to the right and down if needed to avoid crossing.
- Does this help you find a long sequence? Can you draw a tree shape around your diagram with branches that don't cross each other?

Conclusions and Context

Objectives

- To raise awareness that there are unsolved problems and that mathematicians work on identifying new problems and solutions.

Expected Outcomes

- Learners should feel confident that they have explored an unsolved mathematical problem and gain confidence as mathematics students.

Teaching Instructions/Activity

Explain that this problem is an unsolved problem and that in order to have certainty that all numbers will eventually lead to 1 we need an actual proof for it. Mention that many mathematicians have been trying to come up with this proof but have not yet succeeded.

Explain that identifying problems, finding solutions to those problems, and finding possible applications of those solutions to other problems is a big part of the job of mathematicians.

Suggested Guidelines

- Suggested time: any time remaining, but try to keep at least 5 minutes for this.
- You can mention that there are other open problems, called the millennium problems, which even have a prize of USD 1,000,000 for the person who solves one.
- You can encourage students to keep drawing a tree and see how big they can make it including, for example, all numbers up to 50!
- Try to get the students to see how the diagram suggests that all numbers lead to 1.