

Session Guide: The 21 Game

Session Outline

Description

In this session learners will play a counting game that has a clear winning strategy and work towards discovering it. It is an engaging session that guides learners through a journey of discovery, providing opportunities for learners to justify mathematical ideas carefully, and allowing them to extend their creativity by creating variations of the game. The game is quite simple: 2 players take it in turn to count from 1 to 21 with the option of counting 1, 2 or 3 numbers. The player who counts the number 21 loses the game. Learners will have an opportunity to try to beat their teacher (the winning strategy is provided below!), play against each other, and challenge each other with games of their own creation.

Session Objectives

The objectives of this session are to:

- Be able to think strategically when playing the game and understand the importance of strategy in games.
- Gain confidence in their capacity to make conclusions.
- Engage with mathematical ideas through games.

Expected Outcomes

By the end of the session students will have:

- Followed simple rules to play a simple mathematical game.
- Found a strategy to win the game.
- Understood the justification for the winning strategy.
- Created variations of the game and thought about their winning strategies.

Areas involved

- Mathematics and Financial literacy
 - Math Brain Teasers
 - Riddles
 - The Number System



- Life Skills
 - Communication

Activity: Teacher vs Student

Objectives

To understand how the game works.

To recognise that there has to be a winning strategy.

Expected Outcomes

Learners will have observed and played the game enough times to understand the rules.

Teaching Instructions

Explain the game (see below) and demonstrate it with a volunteer learner. Claim to the students that they cannot beat you and ask for volunteers to challenge you to play. Each time you play you should give the learner the option of whether they want to start and it's highly unlikely that they will not provide a winning opportunity as long as you follow the winning strategy (see below). Keep playing against learners until they fully understand the game and make sure you don't give away the winning strategy.

- Rules: 2 players take it in turn to count from 1 to 21 with the option of counting 1, 2 or 3 numbers. The player who counts the number 21 loses the game. For example, player A counts 1, 2, 3, player B counts 4, 5, player A counts 6, 7, player B counts 8, 9, 10, player A counts 11, player B counts 12, 13, 14, player A counts 16, 17, player B counts 18, player A counts 19, 20, and player B is forced to count 21, losing the game.
- Winning Strategy: The player who counts the number 21 is the loser, so in order to win, you need to make sure you are the one who counts up to 20. You need to take control of the game by reaching a multiple of 4, as this means that your opponent will be able to count 1, 2 or 3 more, allowing you to get to the next multiple of 4 by counting 3, 2 or 1 more numbers respectively, eventually reaching the number 20. So, if you are green and have secured 4, you can ensure to get to 20 regardless of what your yellow opponent counts: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21. As such, if your opponent starts the game, since they can count up to 1, 2, or 3, you are always guaranteed to be able to get to 4, and from 4 you are always guaranteed to get to 20 through the multiples of 4.



Suggested Guidelines

- Be flexible with time, it's best to make sure all learners understand the rules of the game and to demonstrate that you can always (or almost if you are not careful) win.
- If you don't feel completely comfortable in the mechanics of the game and want to make sure that you win you can play a few times making the opponent start and sticking to the multiples of 4. However, this could give away the winning strategy.
- If you let the learner decide who starts and you start you will have to find the first possible opportunity to get to a multiple of 4. For example, the learner could count up to 4 and then up to 8, you can then count to 9 and the learner counts 10 and 11, giving you the opportunity to count to 12, reaching a multiple of 4. If the learner makes you start and is following the winning strategy then it's time to move on to the next activity.

Student Instructions

In this session you will be playing the game 21: 2 players take it in turn to count from 1 to 21 with the option of counting 1, 2 or 3 numbers. The player who counts the number 21 loses the game. For example, player A counts 1, 2, 3, player B counts 4, 5, player A counts 6, 7, player B counts 8, 9, 10, player A counts 11, player B counts 12, 13, 14, player A counts 16, 17, player B counts 18, player A counts 19, 20, and player B is forced to count 21, losing the game. Find someone to play with and see if you can find a way to always beat them.

Activity: Playing the Game

Objectives

- To be able to play the game accurately and confidently.
- To start inferring possible winning strategies.

Expected Outcomes

- Learners will have played the game enough times to understand the rules.
- Learners will have discovered that to win, you need to get to 20, most will have discovered that if you get to 16 you can get to 20, some will have discovered that if you get to 12 you can get to 16 and consequently to 20.



Teaching Instructions

Pair the learners so that they can play against each other. After a few games in the same pair mix the pairs so that they play against other learners. You can mix the pairs again after a short while to encourage further interactions between different groups of learners and help them develop a sense of group.

Initially encourage the learners to simply play the game. After some time, encourage them to think about what they have to do to make sure they win. A few minutes before the end of this activity, encourage them to think about what is better, to start or to let the opponent start.

Suggested Guidelines

- Once again be flexible with time.
- If you find that particular learners are discovering the winning strategy, recognise that you know what they are doing but they should keep their strategy to themselves.

Student Instructions

Keep playing the game, against different opponents if possible. While you are playing ask yourself the following questions:

- Is there anything you can do to improve your chances of winning?
- Is there any way to guarantee that you will win?
- IS it better for you to start counting or your opponent to start?

Activity: Towards a Winning Strategy?

Objectives

- To discover a winning strategy.
- Understand the importance of strategy when playing games.

Expected Outcomes

- Learners will have reflected on the intricacies of the game.
- Learners will have collaboratively found a winning strategy.



Teaching Instructions

Lead a discussion with the learners on what is the winning strategy. Start by asking the question “Is it better for you or your opponent to start?”. Don’t give the answer or confirm whether answers suggested are correct.

Ask what number will guarantee a win. All learners will know that if they get to 20 they win. Some might suggest that they need to get to 16. If someone suggests 16 ask them to justify why. If no one suggests 16 ask what number they have to get to so that they are guaranteed to get to 20. Once there is agreement, ask how to guarantee to get to 16 and so on until there is agreement that they need to get to 4.

Once there is agreement that they need to get to 4 in order to win, pose the question about who should start again. Conclude that the game if played strategically the person who starts will always lose.

Suggested Guidelines

- Be flexible with time so that all learners follow and take part in the discussion.
- Play more games against learners to demonstrate each point.
- Try to lead the discussion by asking questions rather than by explaining, let the learners reach conclusions.

Student Instructions

In this game there is a method to guarantee that you can always win. Think about what numbers you need to get to in order to win. How can you make sure that you get to those numbers? Why is this the case? Is it better for you to start or your opponent? Why?

Activity: Creating New Games

Objectives

- To be able to extend the idea by creating variations of the game and defining the winning strategy.
- To be able to discover winning strategies for variations of the game.

Expected Outcomes

- Learners will have created new games, thinking carefully about how each variation leads to a slightly different winning strategy.



- Learners will have tried different variations created by their opponents and discovered how to beat them.

Teaching Instructions

Explain that the game can be changed by either trying to get to a larger number in order to lose or by allowing to count more numbers in each turn, or a combination of both. Demonstrate a variation of the game by making the losing number 24 and ask if the strategy changes at all. Demonstrate the game up to 21 but allowing to count up to 4 numbers rather than 3 in each turn and ask if the strategy is once again different.

- Underlying ideas: This game is not about multiples of 4 but about remainders when dividing by 4 (or modular arithmetic mod 4). In the case where you lose if you count 21, your objective is to get to 20, a multiple of 4, which leaves a remainder of 0 when divided by 4.
 - If you extend the game so that the losing number is 24, you win by getting to 23, which leaves a remainder of 3 when divided by 4, so the winning strategy is to ensure you get to all the numbers that leave a remainder of 3 when divided by 4, i.e. 3, 7, 11, 15, 19, 23. So in this case, in order to guarantee that you get to 3 it is best to start.
 - If you extend the game to allow players to count up to 4 numbers each time, keeping 21 as the losing number, you still need to get to 20. In order to do so, you will need to get to all the multiples of 5 rather than 4, or in other words the numbers that leave a remainder of 0 when divided by 5.
 - Combining both variants is a bit more difficult. If you allow players to count up to n numbers in each turn, then you need to divide by $n+1$. So if the losing number is x , the objective number is $x-1$. So you want to get to the numbers that leave the same remainder as $(x-1)$ when divided by $(n+1)$. If the remainder is 0 then your opponent should start. Otherwise you should start.

Suggested Guidelines

- Encourage students to always justify their strategy.
- If the learners are getting winning strategies quickly then you may want to lead a discussion about how this relates to remainders.
- You may want to split this into 3 different activities, where you consider the first variation only, then the second variation only, and only if the learners are making good progress and you feel confident with the combination of both you propose combining them.



Student Instructions

There are two ways to change this game that leads to interesting differing winning strategies:

1. Play the game where the losing number is 24 rather than 21. How does the game change? Choose a larger number as the losing number, how does it change in that case?
2. Play the game with 21 as the losing number but instead in each turn players can count 1, 2, 3 or 4 numbers. How does the game change in that case? How about if you are allowed to count even more numbers?
3. How about if you lose the game by counting the number 34 and in each turn a player can count 1, 2, 3, 4, or 5 numbers. Is there a winning strategy in this case?

